ON SHORTNESS EXPONENTS OF FAMILIES OF GRAPHS[†]

BY

GÜNTER EWALD

ABSTRACT

Some results on longest circuits on graphs of cell decompositions of closed 2-surfaces are presented.

1. Introduction

Let $v(G)$ denote the number of vertices of a graph G, and let $h(G)$ denote the maximal length of simple circuits in G. If $\mathscr{G}$ is a family of graphs we call $\sigma(\mathscr{G})$ the *shortness exponent* of $\mathscr{G}$ provided

$$\sigma(\mathscr{G}) = \liminf_{G \in \mathscr{G}} \frac{\log h(G)}{\log v(G)}$$

(see Grünbaum-Walther [5]).

It measures the order of magnitude in which $h(G)$ increases as $v(G)$ tends linearly to infinity. One might expect that at least a certain percentage of vertices could always be covered by a longest circuit, which would imply $\sigma(\mathscr{G}) = 1$ for any given family $\mathscr{G}$. This, however, is not true. Grünbaum and Walther have investigated a number of families for which σ is less than one. In fact, it is not even known whether the shortness exponent is always greater than zero. (Grünbaum and Walther [5] conjecture $\sigma \geqq \log 2/\log 3$ for every family of polyhedral graphs.)

In particular, the following families are of interest. Let $\mathscr{G}(g, r)$ denote the family of all 3-connected planar graphs with the following property.

(1) Every face has at most q sides, and every vertex has valence at most r.

We denote the shortness exponent of $\mathscr{G}(q, r)$ by $\sigma(q, r)$. Grünbaum and Walther [5] establish that

† This work has been supported by the National Research Council of Canada.

Received March 21, 1972 and in revised version August 10, 1973

$$\sigma(6,4) \leqq \log 5/\log 7$$
$$\sigma(12,3) \leqq \log 26/\log 27$$
$$\sigma(3,12) \leqq \log 5/\log 7$$

among other results.

We shall, in this paper, be concerned with families of graphs whose shortness exponent is one. Instead of the families $\mathscr{G}(q,r)$ we shall, more generally, consider the families $\mathscr{M}(q,r)$ of all graphs embedded as 1-skeletons in cell complexes whose sets are closed 2-manifolds of arbitrary genus and orientation, such that (1) is satisfied. We set $\sigma(\mathscr{M}(q,r)) = s(q,r)$. Our results are summarized in Theorem 1.1.

THEOREM 1.1. (i) $s(4,4) = 1$

(ii) $s(6,3) = 1$

(iii) $s(3,7) = 1$.

In an earlier paper [2] we have shown that all members of $\mathscr{M}(3,6)$ are Hamiltonian. The *pull over* method developed there is also used here for the proof of Theorem 1.1. (For proof of (i) see Section 2; for proof of (ii) see Section 3; for proof of (iii) see Section 4.)

If p_k is the number of k-sided faces of a graph G of any family $\mathscr{M}(q,r)$, and if v_k is the number of k-valent vertices of G then, by a conclusion from Euler's theorem,

$$\sum_{k \geqq 3} (4-k)(p_k + v_k) = 8(1-g)$$

where g is the genus of the manifold carrying G (compare Grünbaum [4]). Therefore, $\mathscr{M}(4,4)$ consists only of the family $\mathscr{G}(4,4)$ plus the corresponding families on the torus and on the projective plane. In case of the torus, $p_3 + v_3 = 0$, hence $p_3 = v_3 = 0$, and the graphs are Hamiltonian by a result of Altshuler [1]. For $\mathscr{G}(4,4)$ a stronger result can also be established [3]. However, $\mathscr{G}(4,4)$ is not Hamiltonian [6].

J. Kraeft has contributed to the proof of part (iii) of Theorem 1.1.

2. Proof of Theorem 1.1 (i)

We begin by stating an obvious fact.

LEMMA 2.1. *Let every member G of a family $\mathscr{M}(q,r)$ of graphs possess a circuit $H(G)$ with the following property: there exists a constant integer m,* *depending only on $\mathcal{M}(q,r)$, such that every vertex of G has distance at most m from $H(G)$. Then* $s(q,r) = 1$.

We shall use Lemma 2.1 in this section as well as in subsequent sections.

Let G be a member of $\mathcal{G}(4,4)$. If $F^{(0)}$ is a face of G (that is, (i) of the polytope P. in E^3 such that $\text{skel}_1 P = G$, or (ii) of the closed 2-dimensional cell complex having G as skeleton), its three or four boundary edges form a circuit H_0. If $F^{(1)}$ is adjacent to F, that is, has an edge in common with $F^{(0)}$, we *pull* H_0 over $F^{(1)}$ as indicated in Fig. 1, obtaining a circuit H_1 that contains all vertices of $F^{(0)} \cup F^{(1)}$

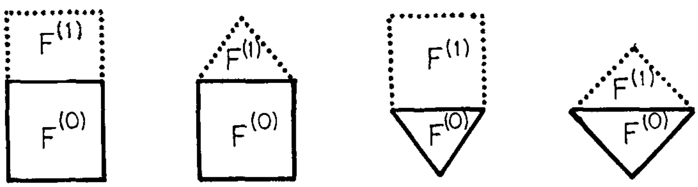

Fig. 1.

Then we choose an appropriate face $F^{(2)}$ adjacent to $F^{(0)}$ or $F^{(1)}$ and change H_1 into a circuit containing all vertices of $F^{(0)} \cup F^{(1)} \cup F^{(2)}$. In this way we build up circuits with an increasing number of vertices. We also apply operations as shown in Fig. 2. In these, we *lose* one vertex and *win* three or two vertices.

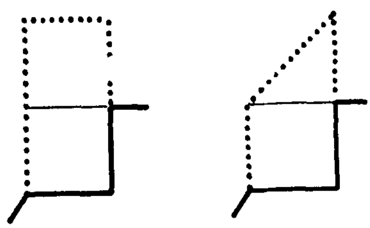

Fig. 2.

Suppose now that the elementary operations described above have been applied in such a way that a maximal number of vertices of G is covered by the circuit H.

LEMMA 2.2. *If a face F of G has no vertex on H then all vertices adjacent to the vertices of F are on H.*

PROOF. If no vertex adjacent to a vertex of F is on H we set $F = F_1$ and choose a face F_2 of G adjacent to F. Clearly, F_2 has no vertex lying on H. If no vertex adjacent to a vertex of F_2 is also on H, we again choose a face $F_3 \neq F_1$ adjacent to F_2. Continuing in this way we eventually find a face F_l which possesses no vertex lying on H, such that a vertex p_1 of G adjacent to a vertex b of F_l is on H. Let a, c be the vertices of F_l adjacent to b (Fig. 3).

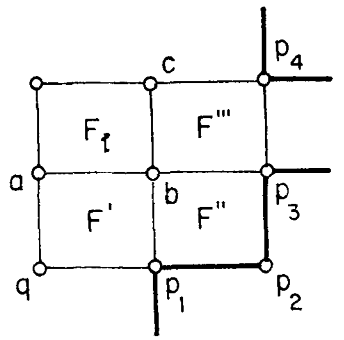

Fig. 3.

If p_1 had valence 3, H would contain an edge of the face F' with a, b, p_1 or c, b, p_3, say a, b, p_1 as vertices. Since a is not on H, the face F' would be a quadrangle with a fourth vertex q. Then, however, H could be prolonged by pulling qp_1 over F', that is, replacing qp_1 by $qabp_1$. This contradicts the maximality of H. So we may assume p_1 to have valence 4 and we denote by F'' the face satisfying $F' \cap F'' = p_1b$. Furthermore, we denote by p_2 the vertex $\neq b$ of F'' adjacent to p_1. Clearly, p_1p_2 is on H. If F'' were a triangle, p_1p_2 could be replaced by the longer path p_1bp_2, a contradiction. So F'' is a quadrangle. We denote its fourth vertex by p_3.

If p_3 were not on H, we could replace p_1p_2 by $p_1bp_3p_2$. Thus p_3 is on H. Let F''' be the face adjacent to F_l and F''. If F''' were a triangle or if its fourth vertex p_4 were not on H, we could apply one of the elementary operations of Fig. 2 and extend H. So F''' is a quadrangle, and its fourth vertex p_4 lies on H. Clearly, p_3p_4 is not on H since otherwise p_3p_4 could be replaced by p_3bcp_4. Therefore, H must contain the edges indicated by heavy lines in Fig. 3.

We apply to p_4 the same arguments as we have applied to p_1. Continuing in this way, we obtain the situation shown in Fig. 4 or 5, depending on whether F_l is a quadrangle or a triangle. In both cases, all vertices adjacent to a vertex of F_l lie on H. Therefore, $l = 1$, and Lemma 2.2 follows.

Clearly, every vertex of G now has distance at most 1 from H. Therefore, by Lemma 2.1, $s(4,4) = 1$.

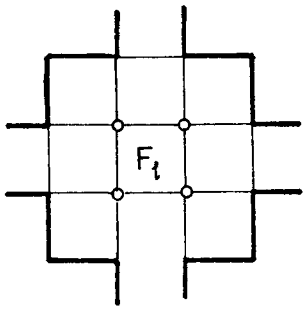

Fig. 4.

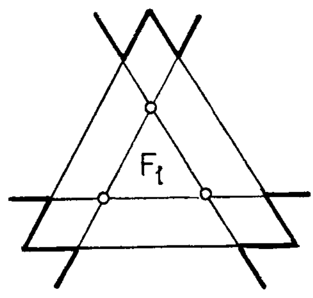

Fig. 5.

3. Proof of Theorem 1.1 (ii)

Let G be a member of $\mathscr{G}(6,3)$. As in Section 2, we let H_0 be the boundary of a face $F^{(0)}$ and pull it over a face $F^{(1)}$ adjacent to $F^{(0)}$, obtaining a circuit passing through all vertices of $F^{(0)} \cup F^{(1)}$. Again, we assume this process has been carried out in an optimal way so as to build up a circuit H in G covering a maximal number of vertices.

LEMMA 3.1. *H contains a vertex of every face of G.*

PROOF. Suppose there exists a face F free of vertices of H. We may choose F in such a way that there exists a facet F' adjacent to F that is not free of vertices of H. If only one side xy of F' were on H, we could replace xy by the remaining sides of F', obtaining a circuit longer than H. So we can assume at least three vertices of F' to lie on H among which is one, say p, adjacent to a vertex a of F. Let b be the vertex of F adjacent to a such that $F \cap F' = ab$. We denote by q the vertex $\neq a$ of F' adjacent to p, and by r the vertex $\neq a$ of F' adjacent to b. F'

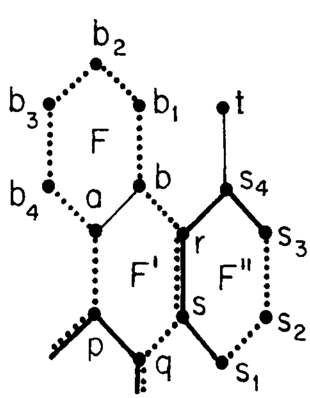

Fig. 6.

cannot be a triangle since otherwise a or b would lie on H. If F' were a pentagon or a quadrangle ($q = r$ in the latter case), H would necessarily contain qr (in case $q \neq r$) or pr (in case $q = r$), and hence could be extended by substituting $pabr$ for pqr or pr, respectively. This contradicts the maximality of H. So let s be the sixth vertex of F'.

We denote the vertices of F by a, $b = b_0, b_1, \cdots, b_n$, where $4 \geqq n \geqq 1$ and b_i is adjacent to b_{i-1}, $i = 1, \cdots, n$.

If s were not on H, then r would not be on H, and we could pull H over F', a contradiction to the maximality of H. So s lies on H. If qs were part of H, we could replace pqs or $pqsr$ (in case r, and hence sr, are also on H) by $pab_nb_{n-1} \cdots b_1brs$ or $pab_nb_{n-1} \cdots b_1br$, respectively, thus winning at least three points and losing at most two. This again contradicts the maximality of H.

Therefore, rs is on H, but qs is not on H. Let F'' be the face of G satisfying $F' \cap F'' = rs$. We denote the vertices of F'' by $r, s = s_0, s_1, \cdots, s_n$, where $4 \geqq n \geqq 2$ and s_i is adjacent to s_{i-1}, $i = 1, \cdots, n$. The same arguments applied to p we now apply to r, obtaining that rs_n is on H but s_nt is not on H, where $t \neq s_{n-1}$ is adjacent to s_n. Thus s_ns_{n-1} is part of H. This implies that F'' is a pentagon or a hexagon; in the former case we set $s_3 = s_2$. If F'' is a hexagon, either s_1s_2 is on H, s_2s_3 is on H, or s_2 is not on H. In any of these cases we can pull back H over F'', replacing $s_2s_1srs_4s_3$, $s_1srs_4s_3s_2$, or $s_1srs_4s_3$ by s_2s_3, s_1s_2, or $s_1s_2s_3$, respectively. Having done this we replace pq by $pab_n \cdots b_1\, brsq$. Altogether we lose at most two vertices and win at least three. This contradicts the maximality of H and we have proved Lemma 3.1.

Now clearly every vertex of G is seen to have distance at most 3 from H. Therefore, by Lemma 2.1, $s(6,3) = 1$.

4. Proof of Theorem 1.1 (iii)

Let now G be a member of $\mathscr{G}(3,7)$. Starting with the boundary of some triangle we build up a circuit by pulling it successively over faces of G, as we have done in the preceding sections for other graphs. Suppose we have obtained a circuit H of maximal possible length in G.

LEMMA 4.1. *If a facet T has no vertex on H then there exists a vertex adjacent to a side of T that is on H.*

PROOF. As in the proof of Theorem 1.1 (i), we may assume that there exists a vertex p of H adjacent to a vertex, say r_1, of T. Suppose none of the vertices

t_1, t_2, t_3 not on T but adjacent to the sides of T, respectively, is on H. Let the r_i and the t_i be numbered as in Fig. 7 ($i=1, 2, 3$). We also introduce T_1, T_2, T_3 as shown in the figure.

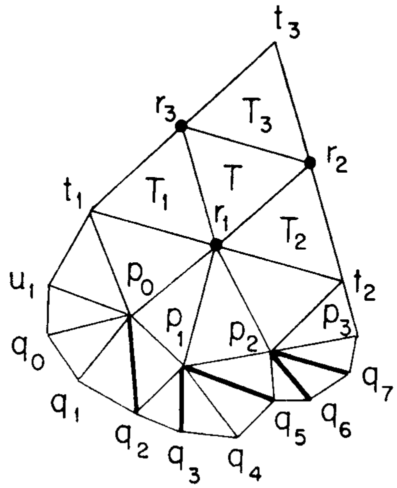

Fig. 7.

First we suppose $p = p_0$ to be adjacent to t_1r_1. Let $p_1 \neq t_1$ be adjacent to p_0r_1 and $u_1 \neq r_1$ be adjacent to t_1p_0. If p_0p_1 were on H we could replace it by $p_0r_1p_1$, a contradiction.

Nor is p_0u_1 on H. Let $p_1, p_3, q_0, q_1, \cdots, q_7$ be introduced as in Fig. 7 where it is assumed that each of the vertices p_0, p_1, p_2 has valence 7. If p_0q_2 is part of H, p_1q_2 cannot be on H since, otherwise, we could replace $p_0q_2p_1$ by $p_0t_1r_1p_1$. But p_1 must lie on H since, otherwise, p_0q_2 could be replaced by $p_0r_1p_1q_2$. If $q_3p_1q_4$ or $q_4p_1q_5$ were a path of H we could replace it by q_3q_4 or q_4q_5, respectively, and extend p_0q_2 as before. p_1p_2 is not on H since it could be extended to $p_1r_1p_2$. So $q_3p_1q_5$ is on H. By similar argument it is shown that p_2 is on H but neither p_2q_5 nor p_2p_3 is on H. Therefore, $q_6p_2q_7$ is on H. Then we can replace $q_6p_2q_7$ by q_6q_7 and p_1q_5 by $p_1r_1p_2q_5$, again a contradiction.

If one of the vertices p_0, p_1, p_2 has valence < 7 we obtain similar contradictions. Thus we find that $q_0p_0q_1$ is a path of H.

Let u_1, u_2, u_3, u_4 be introduced as shown in Fig. 8. We assume again that they all have valence 7; if not, the arguments become easier and can be left out here. Hence we can introduce s_1, $s_2, \cdots, s_{10}$ as shown in Fig. 8.

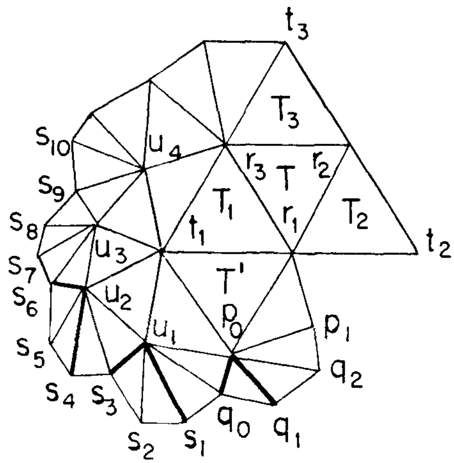

Fig. 8.

If u_1 were not on H, we could replace p_0q_0 by $p_0u_1q_0$. If u_1q_0 were on H we could replace $u_1q_0p_0$ by $u_1t_1r_1p_0$. If $s_1u_1s_2$ or $s_2u_1s_3$ were on H we could replace it by s_1s_2 or s_2s_3, respectively, and extend p_0q_0 to $p_0r_1t_1u_1q_0$. Clearly, u_1u_2 is not on H. Therefore, $s_1u_1s_3$ must be a path of H.

u_2 is on H since, otherwise, u_1s_3 could be replaced by $u_1u_2s_3$. If u_2s_3 were on H, we could replace $q_0p_0q_1$ by q_0q_1 and $u_2s_3u_1$ by $u_2t_1r_3r_1p_0u_1$. None of the paths $s_4u_2s_5$ or $s_5u_2s_6$ is on H since it could be replaced by s_4s_5 or s_5s_6, respectively, so that u_1s_3 were extendable to $u_1t_1u_2s_3$. Nor is u_2u_3 on H. Therefore, $s_4u_2s_6$ is on H.

Clearly, u_3 must also lie on H, but none of the paths $s_7u_3s_8$ or $s_8u_3s_9$ is on H. Suppose u_3s_6 is on H. Then either (i) u_4 is not on H and we replace $u_3s_6u_2$ by $u_3u_4r_3r_1t_1u_2$, or (ii) u_4 is on H. In the second case the same arguments as applied to p_0 show that $s_9u_4s_{10}$ is a path of H. We can replace $s_9u_4s_{10}$ by s_9s_{10}, and replace $u_2s_6u_3$ by $u_2t_1r_1r_3u_4u_3$. Since u_3u_4 is not on H, we conclude that $s_7u_3s_9$ is on H. If u_4 is not on H, we replace u_3s_9 by $u_3t_1u_4s_9$. If u_4, and hence $s_9u_4s_{10}$ (as $q_0p_0q_1$), is on H, we replace $u_3s_9u_4$ by $u_3t_1r_1r_3u_4$.

Therefore, the assumption that p_0 is adjacent to r_1t_1 leads to a contradiction. In fact, no vertex adjacent to a side of T_1, T_2, or T_3 is on H.

Therefore we can replace T by T_1 as shown in Fig. 8. By the same reasoning as used above we see that p_1, u_1 are not on H. Now we replace T by T' and conclude that p_2 is not a vertex of H (Fig. 7). This contradiction proves Lemma 4.1. Hence, by Lemma 4.1 and Lemma 2.1, $s(3,7) = 1$.

REFERENCES

1. A. Altshuler, *Hamiltonian circuits in some maps on the torus*, Discrete Math. **1** (1972), 299–314.
2. G. Ewald, *Hamiltonian circuits in simplicial complexes*, Geometriae Dedicata **2** (1973), 115–125.
3. G. Ewald, *Cut numbers and long circuits in graphs of the family* $\mathscr{G}(4,4)$ (to appear).
4. B. Grünbaum, *Polytopes, graphs, and complexes*, Bull. Amer. Math. Soc. **76** (1970), 1131–1201.
5. B. Grünbaum and H. Walther, *Shortness exponents of families of graphs*, J. Combinatorial Theory (A) **14** (1973), 364–385.
6. C. N. Reynolds, *Circuits upon polyhedra*, Ann. of Math. **33** (1932), 367–372.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA, CANADA
AND
INSTITUT FÜR MATHEMATIK
RUHR-UNIVERSITÄT, BOCHUM, GERMANY